ON THE BEHAVIOR OF THE OSCILLATORY SOLUTIONS OF FIRST OR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Some new results are given concerning the behavior of the oscillatory solutions of first or second order delay differential equations. These results establish that all oscillatory solutions x of a first or second order delay differential equation satisfy $x(t) = O(v(t))$ as $t \longrightarrow \infty$, where v is a nonoscillatory solution of a corresponding first or second order linear delay differential equation. Some applications of the results obtained are also presented.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Oscillation Theory for delay differential equations has grown rapidly during the past two decades. In this time period, several books appeared which presented the current status of this theory and certainly greatly influenced later developments. See the books by Erbe, Kong and Zhang [1], Gopalsamy [2], Györi and Ladas [3], and Ladde, Lakshmikantham and Zhang [5].

In this paper, we deal with the behavior of the oscillatory solutions for first or second order (linear as well as not necessarily linear) delay differential equations. More precisely, we establish that all oscillatory solutions x of a first or second order delay differential equation are such that $x(t) = O(v(t))$ as $t \longrightarrow \infty$, where v is a nonoscillatory solution of a corresponding first or second order linear delay differential equation. The results obtained are heading towards a new direction in the study of the oscillatory solutions of delay differential equations. As far as the authors know, the only result relative to the ones presented here is Theorem 2 in the paper by Huang [4].

Consider the first order linear delay differential equation

$$(\mathrm{E}_1, \delta) \qquad x'(t) + \delta \sum_{k \in K} p_k(t) x(t - \tau_k(t)) = 0 \qquad (\delta = \pm 1)$$

as well as the second order linear delay differential equation

$$(\mathrm{E}_2) \qquad x''(t) + \sum_{k \in K} p_k(t) x(t - \tau_k(t)) = 0,$$

where K *is an initial segment of natural numbers* ($K \neq \varnothing$), p_k *for* $k \in K$ *are nonnegative continuous real-valued functions on the interval* $[0, \infty)$, *and* τ_k *for*

2000 *Mathematics Subject Classification.* 34K11, 34K12, 34K25.

Key words and phrases. Delay differential equation, oscillatory solutions, nonoscillatory solutions.

$k \in K$ are nonnegative continuous real-valued functions on $[0, \infty)$ such that

$$\lim_{t\to\infty}[t - \tau_k(t)] = \infty \quad \textit{for } k \in K.$$

Consider also the linear delay differential equations

$$(\mathrm{E}_1^*) \qquad x'(t) + \sum_{i\in I} p_i(t)x(t-\tau_i(t)) - \sum_{j\in J} p_j(t)x(t-\tau_j(t)) = 0$$

and

$$(\mathrm{E}_2^*) \qquad x''(t) + \sum_{i\in I} p_i(t)x(t-\tau_i(t)) - \sum_{j\in J} p_j(t)x(t-\tau_j(t)) = 0,$$

where *I and J are subsets of K with $I \cap J = \varnothing$ and $I \cup J = K$.* In the case that one of I or J is the empty set, we use the usual convention $\sum_{\varnothing} = 0$. When $I = \varnothing$, the differential equation (E_1^*) coincides with $(\mathrm{E}_1, -1)$, while for $J = \varnothing$ the differential equation (E_1^*) coincides with $(\mathrm{E}_1, +1)$. Moreover, for $I = \varnothing$ the differential equation (E_2^*) takes the form

$$(\widehat{\mathrm{E}}_2) \qquad x''(t) - \sum_{k\in K} p_k(t)x(t-\tau_k(t)) = 0,$$

while in the case where $J = \varnothing$ the differential equation (E_2^*) coincides with (E_2).

Let $t_0 \geq 0$ and set

$$t_{-1} = \min_{k\in K}\min_{t\geq t_0}[t - \tau_k(t)].$$

(Note that $-\infty < t_{-1} \leq t_0$ and that t_{-1} depends on the delays τ_k for $k \in K$ and the initial point t_0.) By a *solution on* $[t_0, \infty)$ of the first order delay differential equation (E_1, δ) or (E_1^*) we mean a continuous real-valued function x defined on the interval $[t_{-1}, \infty)$, which is continuously differentiable on $[t_0, \infty)$ and satisfies (E_1, δ) or (E_1^*) respectively for all $t \geq t_0$. A *solution on* $[t_0, \infty)$ of the second order delay differential equation (E_2) or (E_2^*) is a continuous real-valued function x on the interval $[t_{-1}, \infty)$, which is twice continuously differentiable on $[t_0, \infty)$ and satisfies (E_2) or (E_2^*) respectively for every $t \geq t_0$.

As usual, a continuous real-valued function defined on an interval $[\tau, \infty)$ is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory.*

The first two main results of the paper are Theorems 1 and 2 below. In these theorems, *it will be supposed that the coefficients p_k for $k \in K$ and the delays τ_k for $k \in K$ are such that*

$$(\mathrm{H}) \qquad \sum_{k\in K} p_k(t)\tau_k(t) > 0 \quad \textit{for all } t \geq 0.$$

Clearly, (H) means that, for any $t \geq 0$, there exists at least one index $k \in K$ so that $p_k(t) > 0$ and $\tau_k(t) > 0$.

Theorem 1. *Let v be a nonoscillatory solution of (E_1, δ). Then every oscillatory solution x of (E_1^*) satisfies*

$$(\mathrm{P}) \qquad x(t) = O(v(t)) \quad \textit{as } t \to \infty.$$

Theorem 2. *Let v be a nonoscillatory solution of (E_2). Then every oscillatory solution x of (E_2^*) satisfies (P).*

In order to present the other two main results of our paper, let us consider the (not necessarily linear) delay differential equations

$(\widetilde{\mathrm{E}}_1, \delta)$
$$x'(t) + \sum_{i\in I} p_i(t)x(t-\tau_i(t)) - \sum_{j\in J} p_j(t)x(t-\tau_j(t)) - \delta f(t, x(t)) = 0 \quad (\delta = \pm 1)$$

and

$$(\widetilde{\mathrm{E}}_2) \qquad x''(t) + \sum_{i\in I} p_i(t)x(t-\tau_i(t)) - \sum_{j\in J} p_j(t)x(t-\tau_j(t)) - f(t, x(t)) = 0,$$

where f *is a continuous real-valued function on* $[0,\infty)\times\mathbf{R}$ *with the sign property*

$$zf(t,z) > 0 \quad \textit{for every } t \geq 0 \textit{ and all } z \in \mathbf{R} - \{0\}.$$

In the case where $I = \varnothing$, the differential equations $(\widetilde{\mathrm{E}}_1, \delta)$ and $(\widetilde{\mathrm{E}}_2)$ take, respectively, the forms

$$x'(t) - \sum_{k\in K} p_k(t)x(t-\tau_k(t)) - \delta f(t, x(t)) = 0 \quad (\delta = \pm 1)$$

and

$$x''(t) - \sum_{k\in K} p_k(t)x(t-\tau_k(t)) - f(t, x(t)) = 0.$$

Moreover, when $J = \varnothing$, the differential equations $(\widetilde{\mathrm{E}}_1, \delta)$ and $(\widetilde{\mathrm{E}}_2)$ can respectively be written

$$x'(t) + \sum_{k\in K} p_k(t)x(t-\tau_k(t)) - \delta f(t, x(t)) = 0 \quad (\delta = \pm 1)$$

and

$$x''(t) + \sum_{k\in K} p_k(t)x(t-\tau_k(t)) - f(t, x(t)) = 0.$$

We will consider only such solutions of the delay differential equations $(\widetilde{\mathrm{E}}_1, \delta)$ and $(\widetilde{\mathrm{E}}_2)$, which are defined for all large t. Let $t_0 \geq 0$ and set $t_{-1} = \min_{k\in K}\min_{t\geq t_0}[t - \tau_k(t)]$ $(-\infty < t_{-1} \leq t_0)$. By a *solution on* $[t_0, \infty)$ of $(\widetilde{\mathrm{E}}_1, \delta)$ we mean a continuous real-valued function x defined on the interval $[t_{-1}, \infty)$, which is continuously differentiable on $[t_0, \infty)$ and satisfies $(\widetilde{\mathrm{E}}_1, \delta)$ for every $t \geq t_0$. A *solution on* $[t_0, \infty)$ of $(\widetilde{\mathrm{E}}_2)$ is a continuous real-valued function x on $[t_{-1}, \infty)$, which is twice continuously differentiable on $[t_0, \infty)$ and satisfies $(\widetilde{\mathrm{E}}_2)$ for $t \geq t_0$.

Theorems 3 and 4 below hold *without the assumption* (H) *on the coefficients* p_k *for* $k \in K$ *and the delays* τ_k *for* $k \in K$.

Theorem 3. *Let* v *be a nonoscillatory solution of* (E_1, δ). *Then every oscillatory solution* x *of* $(\widetilde{\mathrm{E}}_1, \delta)$ *satisfies* (P).

Theorem 4. *Let* v *be a nonoscillatory solution of* (E_2). *Then every oscillatory solution* x *of* $(\widetilde{\mathrm{E}}_2)$ *satisfies* (P).

The proofs of our main results, i.e. of Theorems 1-4, will be given in Section 2. In the last section (Section 3), Theorems 1 and 2 will be applied to the special case of first or second order linear autonomous delay differential equations of simplest type.

Before closing this section, we will give some interesting consequences of our main results. Corollaries 1, 2, 3 and 4 below follow immediately from Theorems 1, 2, 3 and 4 respectively. Note that in Corollaries 1 and 2 *it is supposed that the coefficients p_k for $k \in K$ and the delays τ_k for $k \in K$ satisfy* (H), while Corollaries 3 and 4 hold *without this assumption.*

Corollary 1. (i) *If* (E_1, δ) *has a bounded nonoscillatory solution, then all oscillatory solutions of* (E_1^*) *are bounded.*

(ii) *If* (E_1, δ) *has a nonoscillatory solution that tends to zero at* ∞*, then all oscillatory solutions of* (E_1^*) *tend to zero at* ∞.

Corollary 2. (i) *If* (E_2) *has a bounded nonoscillatory solution, then all oscillatory solutions of* (E_2^*) *are bounded.*

(ii) *If* (E_2) *has a nonoscillatory solution that tends to zero at* ∞*, then all oscillatory solutions of* (E_2^*) *tend to zero at* ∞.

Corollary 3. (i) *If* (E_1, δ) *has a bounded nonoscillatory solution, then all oscillatory solutions of* $(\widetilde{\mathrm{E}}_1, \delta)$ *are bounded.*

(ii) *If* (E_1, δ) *has a nonoscillatory solution that tends to zero at* ∞*, then all oscillatory solutions of* $(\widetilde{\mathrm{E}}_1, \delta)$ *tend to zero at* ∞.

Corollary 4. (i) *If* (E_2) *has a bounded nonoscillatory solution, then all oscillatory solutions of* $(\widetilde{\mathrm{E}}_2)$ *are bounded.*

(ii) *If* (E_2) *has a nonoscillatory solution that tends to zero at* ∞*, then all oscillatory solutions of* $(\widetilde{\mathrm{E}}_2)$ *tend to zero at* ∞.

2. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Let v be a nonoscillatory solution on an interval $[t_0, \infty)$, $t_0 \geq 0$, of the delay differential equation (E_1, δ) and let x be an arbitrary oscillatory solution on an interval $[t_0^*, \infty)$, $t_0^* \geq 0$, of the delay differential equation (E_1^*). Without loss of generality, we can suppose that $v(t) \neq 0$ for all $t \geq t_{-1}$, where $t_{-1} = \min_{k \in K} \min_{t \geq t_0} [t - \tau_k(t)]$. (Clearly, $-\infty < t_{-1} < t_0$.) Furthermore, as the negative of a solution of (E_1, δ) is also a solution of the same equation, we may (and do) assume that v is positive on the interval $[t_{-1}, \infty)$. Next, we set $T_0 = \max\{t_0, t_0^*\}$ and $T_{-1} = \min_{k \in K} \min_{t \geq T_0} [t - \tau_k(t)]$. Obviously, $-\infty < T_{-1} < T_0$ and $T_{-1} = \max\{t_{-1}, t_{-1}^*\}$, where $t_{-1}^* = \min_{k \in K} \min_{t \geq t_0^*} [t - \tau_k(t)]$ $(-\infty < t_{-1}^* < t_0^*)$. Moreover, we define

$$y(t) = \frac{x(t)}{v(t)} \quad \text{for } t \geq T_{-1}. \tag{1}$$

For every $t \geq T_0$, we obtain

$$
\begin{aligned}
y'(t) &= \frac{x'(t)v(t) - x(t)v'(t)}{v^2(t)} = \frac{1}{v(t)}\left[x'(t) - \frac{x(t)}{v(t)}v'(t)\right] \\
&= \frac{1}{v(t)}\left[x'(t) - y(t)v'(t)\right] \\
&= \frac{1}{v(t)}\Bigg[-\sum_{i\in I} p_i(t)x(t-\tau_i(t)) + \sum_{j\in J} p_j(t)x(t-\tau_j(t)) + \\
&\qquad + \delta y(t)\sum_{k\in K} p_k(t)v(t-\tau_k(t))\Bigg] \\
&= \frac{1}{v(t)}\Bigg[-\sum_{i\in I} p_i(t)y(t-\tau_i(t))v(t-\tau_i(t)) + \\
&\qquad + \sum_{j\in J} p_j(t)y(t-\tau_j(t))v(t-\tau_j(t)) + \\
&\qquad + \delta y(t)\sum_{i\in I} p_i(t)v(t-\tau_i(t)) + \\
&\qquad + \delta y(t)\sum_{j\in J} p_j(t)v(t-\tau_j(t))\Bigg]
\end{aligned}
$$

and consequently

(2)
$$
\begin{aligned}
\delta v(t)y'(t) = &\sum_{i\in I} p_i(t)\left[y(t) - \delta y(t-\tau_i(t))\right]v(t-\tau_i(t)) + \\
&+ \sum_{j\in J} p_j(t)\left[y(t) + \delta y(t-\tau_j(t))\right]v(t-\tau_j(t))
\end{aligned}
$$

for all $t \geq T_0$.

Now, we will show that y is bounded on the interval $[T_{-1}, \infty)$. Assume, for the sake of contradiction, that y is unbounded on $[T_{-1}, \infty)$. As $-x$ is also an oscillatory solution of (E_1^*) and $-y = (-x)/v$ on $[T_{-1}, \infty)$, we may (and do) suppose that y is unbounded from above. Clearly, y is oscillatory. Thus, we can choose a sufficiently large $T \geq T_0$ so that

(3)
$$y'(T) = 0$$

and

(4)
$$y(T) > |y(t)| \quad \text{for } T_{-1} \leq t < T.$$

By (3), from (2) it follows that

$$
\begin{aligned}
0 &= \sum_{i\in I} p_i(T)\left[y(T) - \delta y(T-\tau_i(T))\right]v(T-\tau_i(T)) + \\
&\quad + \sum_{j\in J} p_j(T)\left[y(T) + \delta y(T-\tau_j(T))\right]v(T-\tau_j(T)),
\end{aligned}
$$

i.e.

$$\sum_{k\in K} p_k(T)\left[y(T) - \delta_k y(T-\tau_k(T))\right] v(T-\tau_k(T)) = 0, \tag{5}$$

where

$$\delta_k = \delta \text{ if } k \in I, \quad \text{and} \quad \delta_k = -\delta \text{ if } k \in J.$$

The proof will be accomplished by proving that (5) is impossible. We first claim that, for any index $k \in K$, it holds

$$y(T) - \delta_k y(T-\tau_k(T)) \geq 0, \text{ if } \tau_k(T) = 0 \tag{6}$$

and

$$y(T) - \delta_k y(T-\tau_k(T)) > 0, \text{ if } \tau_k(T) > 0. \tag{7}$$

To establish our claim, let us consider an arbitrary index $k \in K$. In the case where $\tau_k(T) = 0$, we obtain

$$\begin{aligned} y(T) - \delta_k y(T-\tau_k(T)) &= y(T) - \delta_k y(T) \\ &= \begin{cases} 0, \text{ if } \delta_k = +1 \\ 2y(T), \text{ if } \delta_k = -1 \end{cases} \end{aligned}$$

and hence, since (4) implies that $y(T) > 0$, we have $y(T) - \delta_k y(T-\tau_k(T)) \geq 0$. If $\tau_k(T) > 0$, then from (4) we derive

$$y(T) > |y(T-\tau_k(T))| \geq \delta_k y(T-\tau_k(T))$$

and consequently $y(T) - \delta_k y(T-\tau_k(T)) > 0$. So, our claim is proved. Next, by taking into account assumption (H), we consider an index $k_0 \in K$ such that $\tau_{k_0}(T) > 0$ and

$$p_{k_0}(T) > 0. \tag{8}$$

Because of (7), we have

$$y(T) - \delta_{k_0} y(T-\tau_{k_0}(T)) > 0. \tag{9}$$

Moreover, (6) and (7) imply that

$$y(T) - \delta_k y(T-\tau_k(T)) \geq 0 \quad \text{for all } k \in K. \tag{10}$$

Furthermore, we obviously have

$$p_k(T) \geq 0 \quad \text{for } k \in K \tag{11}$$

and

$$v(T-\tau_k(T)) > 0 \quad \text{for every } k \in K. \tag{12}$$

Finally, the impossibility of (5) follows immediately by using (8), (9), (10), (11) and (12) and so the proof of the theorem is complete.

Proof of Theorem 2. Let v be a nonoscillatory solution on an interval $[t_0, \infty)$, $t_0 \geq 0$, of the delay differential equation (E_2) and assume that x is an oscillatory solution on $[t_0^*, \infty)$, $t_0^* \geq 0$, of the delay differential equation (E_2^*). As $-v$ is also a solution of (E_2), we can restrict ourselves only to the case where v is eventually positive. Furthermore, there is no loss of generality to suppose that v is positive on the whole interval $[t_{-1}, \infty)$, where $t_{-1} = \min_{k\in K}\min_{t\geq t_0}[t - \tau_k(t)]$ $(-\infty < t_{-1} < t_0)$. Let

T_0 and T_{-1} ($-\infty < T_{-1} < T_0$) be defined as in the proof of Theorem 1. Moreover, let y be defined by (1).

We immediately obtain

$$y'(t) = \frac{1}{v(t)}\left[x'(t) - y(t)v'(t)\right] \quad \text{for } t \geq T_0.$$

Thus, for $t \geq T_0$, we derive

$$\begin{aligned}
y''(t) &= \frac{1}{v(t)}\left\{\left[x'(t) - y(t)v'(t)\right]' - \frac{1}{v(t)}\left[x'(t) - y(t)v'(t)\right]v'(t)\right\} \\
&= \frac{1}{v(t)}\left\{\left[x''(t) - y'(t)v'(t) - y(t)v''(t)\right] - y'(t)v'(t)\right\} \\
&= \frac{1}{v(t)}\left[x''(t) - 2y'(t)v'(t) - y(t)v''(t)\right] \\
&= -\frac{2}{v(t)}y'(t)v'(t) + \\
&\quad + \frac{1}{v(t)}\left[-\sum_{i\in I} p_i(t)x(t-\tau_i(t)) + \sum_{j\in J} p_j(t)x(t-\tau_j(t)) + \right. \\
&\qquad \left. + y(t)\sum_{k\in K} p_k(t)v(t-\tau_k(t))\right] \\
&= -\frac{2}{v(t)}y'(t)v'(t) + \\
&\quad + \frac{1}{v(t)}\left[-\sum_{i\in I} p_i(t)y(t-\tau_i(t))v(t-\tau_i(t)) + \right. \\
&\qquad + \sum_{j\in J} p_j(t)y(t-\tau_j(t))v(t-\tau_j(t)) + y(t)\sum_{i\in I} p_i(t)v(t-\tau_i(t)) \\
&\qquad \left. + y(t)\sum_{j\in J} p_j(t)v(t-\tau_j(t))\right]
\end{aligned}$$

and so

$$(13) \qquad \begin{aligned} v(t)y''(t) + 2y'(t)v'(t) &= \sum_{i\in I} p_i(t)\left[y(t) - y(t-\tau_i(t))\right]v(t-\tau_i(t)) + \\ &\quad + \sum_{j\in J} p_j(t)\left[y(t) + y(t-\tau_j(t))\right]v(t-\tau_j(t)) \end{aligned}$$

for every $t \geq T_0$.

Our purpose is to prove the boundedness of y on the interval $[T_{-1}, \infty)$. For the sake of contradiction, we will suppose that y is unbounded on this interval. Since $-y = (-x)/v$ on $[T_{-1}, \infty)$ and $-x$ is also an oscillatory solution of (E_2^*), we may (and do) assume that y is unbounded from above. Hence, as y is oscillatory, there exists a sufficiently large $T \geq T_0$ such that (3) and (4) are satisfied, and

$$(14) \qquad y''(T) \leq 0.$$

In view of (3) and (14), from (13) we obtain

$$\begin{aligned} 0 \geq & \sum_{i\in I} p_i(T)\,[y(T)-y(T-\tau_i(T))]\,v(T-\tau_i(T)) + \\ & + \sum_{j\in J} p_j(T)\,[y(T)+y(T-\tau_j(T))]\,v(T-\tau_j(T)). \end{aligned}$$

By setting

$$\epsilon_k = +1 \quad \text{if } k\in I, \quad \text{and} \quad \epsilon_k = -1 \quad \text{if } k\in J,$$

the last inequality can be written

$$\sum_{k\in K} p_k(T)\,[y(T)-\epsilon_k y(T-\tau_k(T))]\,v(T-\tau_k(T)) \leq 0. \tag{15}$$

Finally, following the same arguments as in the proof of Theorem 1 with ϵ_k in place of δ_k (for any $k \in K$), we can show that (15) is impossible. This contradiction completes the proof of the theorem.

Proof of Theorem 3. Let v be a nonoscillatory solution on an interval $[t_0,\infty)$, $t_0 \geq 0$, of (E_1,δ) and consider an arbitrary oscillatory solution x on an interval $[\widetilde{t}_0,\infty)$, $\widetilde{t}_0 \geq 0$, of the delay differential equation $(\widetilde{\mathrm{E}}_1,\delta)$. As in the proof of Theorem 1, we may (and do) assume that v is positive on the whole interval $[t_{-1},\infty)$, where $t_{-1} = \min\limits_{k\in K}\min\limits_{t\geq t_0}[t-\tau_k(t)]$ $(-\infty < t_{-1} \leq t_0)$. Set $T_0 = \max\{t_0,\widetilde{t}_0\}$ and $T_{-1} = \min\limits_{k\in K}\min\limits_{t\geq T_0}[t-\tau_k(t)]$ $(-\infty < T_{-1} \leq T_0)$. It is obvious that $T_{-1} = \max\{t_{-1},\widetilde{t}_{-1}\}$, where $\widetilde{t}_{-1} = \min\limits_{k\in K}\min\limits_{t\geq \widetilde{t}_0}[t-\tau_k(t)]$ $(-\infty < \widetilde{t}_{-1} \leq \widetilde{t}_0)$. Moreover, we define the function y by (1).

We obtain for $t \geq T_0$

$$\begin{aligned} y'(t) &= \frac{1}{v(t)}\,[x'(t)-y(t)v'(t)] \\ &= \frac{1}{v(t)}\Bigg[-\sum_{i\in I} p_i(t)x(t-\tau_i(t)) + \sum_{j\in J} p_j(t)x(t-\tau_j(t)) + \\ &\qquad + \delta f(t,x(t)) + \delta y(t)\sum_{k\in K} p_k(t)v(t-\tau_k(t))\Bigg] \end{aligned}$$

and so we can easily find

$$\begin{aligned} \delta v(t)y'(t) = {} & f(t,x(t)) + \\ & + \sum_{i\in I} p_i(t)\,[y(t)-\delta y(t-\tau_i(t))]\,v(t-\tau_i(t)) + \\ & + \sum_{j\in J} p_j(t)\,[y(t)+\delta y(t-\tau_j(t))]\,v(t-\tau_j(t)) \end{aligned} \tag{16}$$

for every $t \geq T_0$.

Now, assume, for the sake of contradiction, that y is unbounded on $[T_{-1},\infty)$. We observe that $-x$ is an oscillatory solution of an equation of the form $(\widetilde{\mathrm{E}}_1,\delta)$ with $\overline{f}$ in place of f, where $\overline{f}(t,z) = -f(t,-z)$ for $(t,z) \in [0,\infty)\times\mathbf{R}$. Clearly, $\overline{f}$ is subject to the same conditions as f. So, since $-y = (-x)/v$ on $[T_{-1},\infty)$, we can confine our discussion only to the case where y is unbounded from above. Hence, by taking into account the oscillatory character of y, we can consider a sufficiently large $T \geq T_0$ such that (3) and (4) are satisfied. Then, by (3), from (16) it follows easily that

$$f(T,x(T)) + \sum_{k\in K} p_k(T)\left[y(T) - \delta_k y(T-\tau_k(T))\right] v(T-\tau_k(T)) = 0, \tag{17}$$

where δ_k for $k \in K$ are defined as in the proof of Theorem 1. In view of (4), we must have $y(T) > 0$ and so $x(T)$ is also positive. Thus, because of the sign property of the function f, we always have

$$f(T,x(T)) > 0. \tag{18}$$

Furthermore, as in the proof of Theorem 1, we conclude that (6) and (7) are fulfilled for any index $k \in K$ and consequently (10) holds. On the other hand, (11) and (12) are obviously satisfied. Finally, by using (10), (11), (12) and (18), we can immediately verify that (17) is impossible. So, our proof is complete.

Proof of Theorem 4. Let v be a nonoscillatory solution on $[t_0,\infty)$, $t_0 \geq 0$, of (E_2) and suppose that x is an oscillatory solution on $[\widetilde{t}_0,\infty)$, $\widetilde{t}_0 \geq 0$, of the delay differential equation $(\widetilde{\mathrm{E}}_2)$. As in the proof of Theorem 2, we assume that v is positive on the whole interval $[t_{-1},\infty)$, where $t_{-1} = \min_{k\in K}\min_{t\geq t_0}[t-\tau_k(t)]$ $(-\infty < t_{-1} \leq t_0)$. Define T_0 and T_{-1} $(-\infty < T_{-1} \leq T_0)$ as in the proof of Theorem 3. Moreover, define y by (1).

As in the proof of Theorem 2, we obtain for $t \geq T_0$

$$\begin{aligned} y''(t) &= \frac{1}{v(t)}\left[x''(t) - 2y'(t)v'(t) - y(t)v''(t)\right] \\ &= -\frac{2}{v(t)}y'(t)v'(t) + \\ &\quad + \frac{1}{v(t)}\Bigg[-\sum_{i\in I} p_i(t)x(t-\tau_i(t)) + \sum_{j\in J} p_j(t)x(t-\tau_j(t)) + \\ &\qquad + f(t,x(t)) + y(t)\sum_{k\in K} p_k(t)v(t-\tau_k(t))\Bigg] \end{aligned}$$

and so we can easily derive

$$\begin{aligned} v(t)y''(t) + 2y'(t)v'(t) &= f(t,x(t)) + \\ &+ \sum_{i\in I} p_i(t)\left[y(t) - y(t-\tau_i(t))\right] v(t-\tau_i(t)) \\ &+ \sum_{j\in J} p_j(t)\left[y(t) + y(t-\tau_j(t))\right] v(t-\tau_j(t)) \end{aligned} \tag{19}$$

for all $t \geq T_0$.

Assume now that y is unbounded on the interval $[T_{-1},\infty)$. Since $-y = (-x)/v$ on $[T_{-1},\infty)$ and $-x$ is also an oscillatory solution of an equation of the form $(\widetilde{\mathrm{E}}_2)$ with $\overline{f}$ in place of f, where $\overline{f}(t,z) = -f(t,-z)$ for $(t,z) \in [0,\infty)\times\mathbf{R}$, we may (and do) suppose that y is unbounded from above. Thus, there exists a sufficiently large

$T \geq T_0$ such that (3), (4) and (14) hold. By (3) and (14), from (19) it follows easily that

$$f(T, x(T)) + \sum_{k \in K} p_k(T)\,[y(T) - \epsilon_k y(T - \tau_k(T))]\, v(T - \tau_k(T)) \leq 0, \tag{20}$$

where ϵ_k for $k \in K$ are defined as in the proof of Theorem 2. By the same technique as in the proof of Theorem 3 with ϵ_k in place of δ_k (for $k \in K$), we can arrive at the contradiction that (20) is impossible. This contradiction completes the proof of the theorem.

3. APPLICATIONS

In this section, we will apply Theorems 1 and 2 to the special case of first or second order linear autonomous delay differential equations. Our interest will be concentrated on the simple case of first or second order linear autonomous delay differential equations with one or two delays.

Consider the first order linear autonomous delay differential equations

$$x'(t) + px(t - \tau) = 0, \tag{D_1}$$

$$x'(t) - px(t - \tau) = 0, \tag{D_2}$$

$$x'(t) + px(t - \tau) + qx(t - \sigma) = 0, \tag{D_3}$$

$$x'(t) - px(t - \tau) - qx(t - \sigma) = 0, \tag{D_4}$$

$$x'(t) + px(t - \tau) - qx(t - \sigma) = 0 \tag{D_5}$$

as well as the second order linear autonomous delay differential equations

$$x''(t) + px(t - \tau) = 0, \tag{D_6}$$

$$x''(t) - px(t - \tau) = 0, \tag{D_7}$$

$$x''(t) + px(t - \tau) + qx(t - \sigma) = 0, \tag{D_8}$$

$$x''(t) - px(t - \tau) - qx(t - \sigma) = 0, \tag{D_9}$$

$$x''(t) + px(t - \tau) - qx(t - \sigma) = 0, \tag{D_{10}}$$

where p, q, τ *and* σ *are positive constants.* As it is natural, *it will be supposed that* $\tau \neq \sigma$.

By applying Theorem 1, we immediately obtain the following results:

Let v be a nonoscillatory solution of (D_1) *or* (D_2). *Then all oscillatory solutions x of* (D_1) *and* (D_2) *satisfy* (P), *i.e.* $x(t) = O(v(t))$ *as* $t \to \infty$.

Let v be a nonoscillatory solution of (D_3) *or* (D_4). *Then all oscillatory solutions x of* (D_3), (D_4) *and* (D_5) *satisfy* (P).

Similarly, an application of Theorem 2 leads to the next results:

Let v be a nonoscillatory solution of (D_6). *Then all oscillatory solutions x of* (D_6) *and* (D_7) *satisfy* (P).

Let v be a nonoscillatory solution of (D_8). *Then all oscillatory solutions x of* (D_8), (D_9) *and* (D_{10}) *satisfy* (P).

It is known that the existence of a nonoscillatory solution of a linear autonomous delay differential equation is guaranteed by the existence of a real root of its characteristic equation. So, the above results can be used to obtain new criteria on the behavior of the oscillatory solutions of the delay differential equations (D_1)-(D_{10}).

REFERENCES

[1] L. H. ERBE, Q. KONG AND B. G. ZHANG, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, Inc., New York, 1995.

[2] K. GOPALSAMY, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic Publishers, Dordrecht, 1992.

[3] I. GYÖRI AND G. LADAS, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[4] Q. HUANG, Comparison theorems for first order retarded functional differential equations and their applications, *Differential Integral Equations* **5** (1991), 1101-1112.

[5] G. S. LADDE, V. LAKSHMIKANTHAM AND B. G. ZHANG, *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, Inc., New York, 1987.

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